5453: **Proposed by D. M. Bătinetu-Giurgiu**, **Matei Basarab National College**, **Bucharest**, **Romania and Neculai Stanciu**,“**George Emil Palade**” **School**, **Buzău**, **Romania**

If $a,b,c \in (0,1)$ or $a,b,c \in (1,\infty)$ and m,n are positive real numbers, then prove that

$$\frac{\log_a b + \log_b c}{m + n\log_a c} + \frac{\log_b c + \log_c a}{m + n\log_b a} + \frac{\log_c a + \log_a b}{m + n\log_c b} \geq \frac{6}{m+n}$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA.**

Let $x := \log_a b = \frac{\ln b}{\ln a}, y := \log_b c = \frac{\ln c}{\ln b}, z := \log_c a = \frac{\ln a}{\ln c}$ and $k := \frac{n}{m} > 0.$

Then $\log_a c = \log_a b \cdot \log_b c = xy, \log_b a = yz, \log_c b = zx, xyz = 1,\ x,y,z > 0$

(because $sign(\ln a) = sign(\ln b) = sign(\ln c)$) and original inequality becomes

$$\sum_{cyc} \frac{x+y}{1 + \frac{n}{m}\cdot\frac{1}{z}} \geq \frac{6}{1+\frac{n}{m}} \Leftrightarrow \sum_{cyc}\frac{x+y}{1+kxy} \geq \frac{6}{1+k} \Leftrightarrow$$

$$\sum_{cyc}\left(\frac{x+y}{1+kxy} + z\right) \geq \frac{6}{1+k} + x + y + z \Leftrightarrow \sum_{cyc}\frac{x+y+z+kxyz}{1+kxy} \geq \frac{6}{1+k} + x + y + z \Leftrightarrow$$

(**1**) $$\sum_{cyc}\frac{1}{1+kxy} \geq \frac{6+(k+1)(x+y+z)}{(1+k)(x+y+z+kxyz)}.$$

Denoting $s := x+y+z,\ p := xy+yz+zx$ and taking in account that $xyz = 1$ we obtain $\sum_{cyc}\frac{1}{1+kxy} = \frac{1}{k^3+1+kp+k^2s}\sum_{cyc}(1+kyz)(1+kzx) = \frac{3+2kp+k^2s}{k^3+1+kp+k^2s}$

and inequality (**1**) becomes

(**2**) $$\frac{3+2kp+k^2s}{k^3+1+kp+k^2s} \geq \frac{6+(k+1)s}{(1+k)(s+k)}$$

Also note that $s = x+y+z \geq 3\sqrt[3]{xyz} = 3, p = xy+yz+zx \geq 3\sqrt[3]{x^2y^2z^2} = 3,$

$s^2 = (x+y+z)^2 \geq 3(xy+yz+zx) = 3p$ and

$p^2 = (xy+yz+zx)^2 \geq 3xyz(x+y+z) = 3s.$

Hence, $\sqrt{3s} \leq p \leq \frac{s^2}{3} \Leftrightarrow \sqrt{3p} \leq s \leq \frac{p^2}{3}$ and $s,p \geq 3.$

We have (**2**)$\Leftrightarrow$ $(3+2kp+k^2s)(k+1)(s+k) - (k^2s+kp+k^3+1)(6+(k+1)s) \geq 0 \Leftrightarrow$

$(2k+2+kp(k+1)-6k^2)s + 3k + 2k^2p + 2k^3p - 6kp + 3k^2 - 6k^3 - 6 \geq 0.$

Since $p \geq 3$ then $2k+2+kp(k+1)-6k^2 \geq 2k+2+3k^2+3k-6k^2 = (3k+1)(2-k).$

If $k \in (0,2]$ *then* $2k+2+k^2p+kp-6k^2 \geq 0$ *and, therefore,*

$(2k+2+kp(k+1)-6k^2)s \geq (2k+2+k^2p+kp-6k^2)\sqrt{3p}.$

Thus, suffice to prove inequality

$(2k+2+kp(k+1)-6k^2)\sqrt{3p} + 3k - 2kp(3-k-k^2) + 3k^2 - 6k^3 - 6 \geq 0 \Leftrightarrow$

(**3**) $(2k+2+k(k+1)p-6k^2)\sqrt{3p} + 3k - 2kp(3-k-k^2) + 3k^2 - 6k^3 - 6 \geq 0$

for $p \geq 3.$

By replacing p in inequality (**3**) with $3t^2$ we can rewrite (3) as

$3t(2k+2+k(k+1)\cdot 3t^2 - 6k^2) + 3k - 2k(3-k-k^2)\cdot 3t^2 + 3k^2 - 6k^3 - 6 \geq 0 \Leftrightarrow$

$3(t-1)(3k(k+1)t^2 + k(k+3)(2k-1)t + (k+1)(2k^2-3k+2)) \geq 0.=$

Since $t \geq 1 \Leftrightarrow p \geq 3$ suffice to prove that

$3k(k+1)t^2 - k(k+3)(1-2k)t + (k+1)(2k^2-3k+2) \geq 0$ for $t \geq 1.$

Noting that abscise of vertex $\frac{k(k+3)(1-2k)}{6k(k+1)} < 1$

(because $6k(k+1) - k(k+3)(1-2k) = k(11k+2k^2+3) > 0)$ then

for $t \geq 1$ we have

$3k(k+1)t^2 - k(k+3)(1-2k)t + (2k^2-3k+2) \geq$

$3k(k+1) \cdot 1^2 - k(k+3)(1-2k) \cdot 1 + (2k^2-3k+2) =$

$2k^3 + 10k^2 - 3k + 2 > 2 - 3k(1-k) \geq 2 - \frac{3}{4} > 0.$

Consider now case $k > 2$.

Coming back to inequality (**2**) we note that $\frac{3+2kp+k^2s}{k^3+1+k^2p+ks}$ increase in $p > 0$

because $\frac{3+2kp+k^2s}{k^3+1+k^2p+ks} = 2 - \frac{k^2s+2k^3-1}{k^2s+kp+k^3+1}$ and

$2k^3+k^2s-1 \geq 2k^3+3k^2-1 > 0$ for $k > 2$.

Therefore $\frac{3+2kp+k^2s}{k^3+1+k^2p+ks} \geq \frac{3+2kp+k^2s}{k^3+1+k^2p+ks} \geq \frac{3+2k \cdot 3+k^2s}{k^3+1+k^2 \cdot 3+ks} = \frac{3+6k+k^2s}{k^3+1+3k^2+ks}$

and $\frac{3+6k+k^2s}{k^3+1+3k^2+ks} - \frac{6+(k+1)s}{(1+k)(s+k)} = \frac{(s-3)(s(k^3-k)+3k^2-k+2)}{(k+1)(k+s)(ks+3k^2+k^3+1)} \geq 0$

because $s(k^3-k)+3k^2-k+2 > 0$ for $s \geq 3$ and $k > 2$.

Remark1. In the case $k = 2$ the inequality became simple:

$\sum_{cyc} \frac{x+y}{1+2xy} \geq 2 \Leftrightarrow \frac{3+4p+4s}{9+2p+4s} \geq \frac{6+3s}{3(s+2)} = 1 \Leftrightarrow$

$3+4p+4s-(9+2p+4s) \geq 0 \Leftrightarrow 2(p-3) \geq 0.$

Remark2. One simple modification of original inequality.

Prove that $\frac{\log_a b+\log_b c}{m+n\log_c a} + \frac{\log_b c+\log_c a}{m+n\log_a b} + \frac{\log_c a+\log_a b}{m+n\log_b c} \geq \frac{6}{m+n}.$

In x,y,z notation it equivalent to $\sum_{cyc} \frac{x+y}{m+nz} \geq \frac{6}{m+n}.$

By Cauchy $\sum_{cyc} \frac{x+y}{m+nz} = \sum_{cyc} \frac{(x+y)^2}{(x+y)(m+nz)} \geq \frac{4(x+y+z)^2}{\sum_{cyc}(x+y)(m+nz)}.$

Thus remains to prove $\frac{4(x+y+z)^2}{\sum_{cyc}(x+y)(m+nz)} \geq \frac{6}{m+n} \Leftrightarrow$

$2(m+n)(x+y+z)^2 \geq 3\sum_{cyc}(x+y)(m+nz) = 6m(x+y+z)+6n(xy+yz+zx).$

Let $s := x+y+z, p := xy+yz+zx.$ Then latter inequality becomes

$(m+n)s^2 \geq 3ms+3np.$

Since $s \geq 3$ and $s^2 \geq 3p$ then $(m+n)s^2 \geq 3ms+3np.$